Portfolio Management in the Presence of Stochastic Volatility

by

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Abstract

Merton in 1969 considered how an investor must decide how much to consume and how to allocate their wealth between risky and sure assets in order to maximise their expected lifetime utility. This classic problem is sometimes called Merton's Problem.

In this project, we attempt to replace the stochastic differential equation for the risky asset, which Merton assumed to have constant volatility, with one that has volatility of a stochastic nature. Then we will attempt to solve this modified Merton problem for two cases:

1. Finite Time Horizon (Chapter 2).
2. Infinite Time Horizon (Chapter 3).

We will apply numerical methods to solve Merton's optimisation problem and analyse how each parameter affects the portfolio. (Chapter 4), (Chapter 5).

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Contents

Chapter 1

Introduction

Merton in the paper [1] examined the portfolio optimisation problem for an individual whose income is generated by capital gains on assets satisfying geometric Brownian motion. He later in 1971 extended the idea to more general utility functions, as well as considering income generated from non-capital gain sources [2]. Merton solved this optimal portfolio problem in a market with a sure asset and a risky asset, the price of which follows a geometric Brownian motion with dynamics:

$$dP_t = aP_t dt + bP_t dW_t \tag{1.1}$$

where a, b are constants.

However, analysis carried out by Scott [4] has shown that the volatility is in reality not constant. Having this in mind, in order for us to get a more accurate representation of the dynamics of the asset, we must take into account a randomly changing volatility.

Furthermore[1], a well-known fact noticed first by Black [3] is the negative correlation between volatility and current stock prices. Known as the *Leverage Effect*, this observation strengthens the link between volatility and current stock prices. A working paper by Stephen Figlewski & Xiaozu Wang [10] provides many references to this phenomenon.

Volatility Persistence is another well-documented fact, where volatility is observed to have a memory. This reinforces the link between volatility and past prices. A paper by Mathieu Rosenbaum [11] provides further references. The aim of this project is to extend the Merton problem so it takes into

[1]The topics we discuss here are an expansion of the ideas discussed in [6].

account randomly changing volatility. We shall choose a model that captures at least some of the above characteristics, unlike equation (1.1).

As mentioned by Davis [12], stochastic volatility models divide into two broad classes: 'single-factor' and 'multi-factor' models. In the former, the original Brownian motion W_t continues to be the only source of randomness, whereas in the latter further Brownian motions or other random elements are introduced. In this paper, we will empathise mainly on 'single-factor' models. We refer the reader to [12], which discusses 'multi-factor' models in detail. In the next section, we will briefly discuss the simplest case of the 'single-factor' models, the so-called Level Dependent Volatility model.

1.1 Level Dependent Volatility

In the level dependent volatility model, the price process takes the form:

$$dP_t = rP_t dt + \sigma(P_t)P_t dW_t \tag{1.2}$$

where σ is a Lipschitz continuous function.

Several well-known forms are the CEV model [14] and the implied tree models by Derman & Kani [15], Dupire [16] or Rubinstein [17]. However analysis done by Dumas, Fleming & Whaley [18] led them to conclude that the deterministic volatility framework could be generalised, suggesting that volatility may be related to past changes in the underlying. Hobson & Rogers proposed a new class of models [5], which was inspired by the GARCH framework. The unique aspect of this model is that it defines the volatility in terms of the historical performance of the stock price. The distinct advantage this model has over the traditional stochastic volatility models is that they do not introduce any new source of randomness, thus maintaining market completeness. Moreover, as we are looking at the past performance of the stocks we are able to incorporate economic trends easily without the need to introduce other variables. In the next section, we will briefly look at the dynamics behind the Hobson & Rogers model.

1.2 Hobson & Rogers Model

Let us briefly review the Hobson & Rogers model [5]. Firstly, denote the discounted log price process Z_t by:

$$Z_t = \log(P_t e^{-rt}), \tag{1.3}$$

where P_t is the stock price at time t.

Definition 1.2.1. *The first order offset function S_t, is defined as:*

$$S_t = \int_0^\infty \lambda e^{-\lambda u}(Z_t - Z_{t-u})du, \tag{1.4}$$

where λ is a positive constant which expresses the discounting rate of past information. [2]

Moreover, given W_t is a standard Brownian motion, we assume Z_t satisfies the SDE:

$$dZ_t = \mu(S_t)dt + \sigma(S_t)dW_t \tag{1.5}$$

where $\sigma(.)$ and $\mu(.)$ are Lipschitz functions, with $\sigma(.)$ being strictly positive.

Lemma 1.2.1. *(Z_t, S_t) forms a Markov process. The offset function S_t satisfies the SDE:*

$$dS_t = dZ_t - \lambda S_t dt \tag{1.6}$$

For the proof of this Lemma, we will refer the reader to the Hobson & Rogers paper [5]. Equations (1.5) and (1.6) give us:

$$dS_t = [\mu(S_t) - \lambda S_t]dt + \sigma(S_t)dW_t \tag{1.7}$$

In my project, I propose to solve Merton's problem with the asset price following the Hobson & Rogers model. This will be accomplished by using the method of dynamic programming. As stated by Benth [19], a major drawback with the dynamic programming approach is that the allocation strategies must depend on the volatility. As volatility is not directly observable in the market, it is practically impossible to follow portfolio rules if volatility levels are explicitly taken into account. However, in our context the Hobson & Rogers model defines the volatility in terms of stock prices, which the investor can observe. This means our model is partially observable, thus the above drawback is effectively made redundant.

[2]The paper by Hobson & Rogers actually introduces a more general model with m offsets. This is achieved by defining $S_t^{(m)}$ by $S_t^{(m)} = \int_0^\infty \lambda e^{-\lambda u}(Z_t - Z_{t-u})^m du$. In this paper for simplicity, we will only consider 1 offset.

1.3 Stochastic Control

Let us first briefly introduce the concept of optimal control. Optimal control theory is a generalisation of the calculus of variations optimisation method for deriving control policies. Classical calculus of variations developed from the brachistochrone problem posed by the Swiss mathematician Johann Bernoulli. An in-depth review of this topic can be found in [20]. However, classical calculus of variations cannot be directly applied to trajectories that are stochastic processes. The reason for this is that stochastic trajectories will be non-smooth, violating admissibility conditions [21]. Instead, we turn to the method of dynamic programming in continuous time, often called *Stochastic Control*. Unlike classical control theory, a stochastic control takes into account uncertainties, such as random noises and disturbances, which make it ideal for tackling Merton's problem. This has been discussed in detail by Duffie [7] and to some extent by Musiela & Zariphopoulou [9]. A key notion in stochastic control theory is the *Hamilton Jacobi Bellman equation*. In the next two sections I will outline how to derive the *Hamilton Jacobi Bellman equation* for both finite and infinite horizons. The following sections are based largely on the work of [7] & [23].

1.3.1 Finite Horizon

In this section, we will derive the stochastic version of the *Hamilton Jacobi Bellman equation* for the finite horizon stochastic optimisation problem. We begin by formulating the problem as Merton [1]:

$$V(t,h) = \sup_{c_t} \mathbb{E}_t \left\{ \int_t^T e^{-\rho\tau} u(c_\tau) d\tau + U(H_T) \right\} \qquad \rho > 0 \qquad (1.8)$$

such that $dH_t = j(t, H_t, c_t)dt + k(t, H_t, c_t)dW_t$

- t denotes the initial time and h denotes the corresponding state.
- H_T is an $\mathcal{F}_T$ measurable nonnegative random variable describing the wealth at time T.
- $\mathbb{E}_t$ denotes the *expectation* given the information available upto time t.
- $V(t,h)$ is the value function which measures the value of the state at time t, with a state variable h.

- c_t is our control variable, it is an adapted nonnegative consumption rate process with $\int_t^T c_\tau d\tau < \infty$ almost surely.
- The *utility of intermediate consumption* $u : \mathbb{R}^+ \times [t,T] \to \mathbb{R}$ is continuous and $\forall \tau \in [t,T]$, $u : \mathbb{R}^+ \to \mathbb{R}$ is increasing and concave, with $u(0) = 0$.
- The *utility of terminal wealth* $U : \mathbb{R}^+ \to \mathbb{R}$ is increasing and concave, with $U(0) = 0$.

Now following a route similar to Dorfman [22], we can approximate V by keeping c constant in the interval $[t, t+dt]$, then afterwards assuming maximising behaviour:

$$V(t,h) = \sup_{c_t} \mathbb{E}_t \left\{ e^{-\rho t} u(c_t)dt + V(t+dt, H_{t+dt}) \right\} \tag{1.9}$$

The previous equation is also known as *The Bellman principle of optimality* [13]. An application of Itô's lemma yields:

$$\begin{aligned} dV_t &= \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial h}dH_t + \frac{1}{2}\frac{\partial^2 V}{\partial h^2}d\langle H\rangle_t \\ &= \left[\frac{\partial V}{\partial t} + \frac{\partial V}{\partial h}j(t,H_t,c_t) + \frac{1}{2}\frac{\partial^2 V}{\partial h^2}k(t,H_t,c_t)^2\right]dt + \left[\frac{\partial V}{\partial h}k(t,H_t,c_t)\right]dW_t \end{aligned} \tag{1.10}$$

Remembering that $dV_t = V(t+dt, H_{t+dt}) - V(t, H_t)$ we get:

$$\begin{aligned} V(t+dt, H_{t+dt}) = V(t,H_t) &+ \left[\frac{\partial V}{\partial t} + \frac{\partial V}{\partial h}j(t,H_t,c_t) + \frac{1}{2}\frac{\partial^2 V}{\partial h^2}k(t,H_t,c_t)^2\right]dt \\ &+ \left[\frac{\partial V}{\partial h}k(t,H_t,c_t)\right]dW_t \end{aligned} \tag{1.11}$$

Hence equation (1.9) becomes:

$$\begin{aligned} V(t,h) = \sup_{c_t} \mathbb{E}_t \Bigg\{ e^{-\rho t}u(c_t)dt + V(t,H_t) + \Bigg[\frac{\partial V}{\partial t} + \frac{\partial V}{\partial h}j(t,H_t,c_t) \\ + \frac{1}{2}\frac{\partial^2 V}{\partial h^2}k(t,H_t,c_t)^2\Bigg]dt + \left[\frac{\partial V}{\partial h}k(t,H_t,c_t)\right]dW_t \Bigg\} \end{aligned} \tag{1.12}$$

Since $\mathbb{E}_t[W_t] = 0$, we can simplify the previous equation to the *Hamilton Jacobi Bellman equation*:

$$\sup_c \left\{ e^{-\rho t}u(c) + \frac{\partial V}{\partial t} + \frac{\partial V}{\partial h}j(t,h,c) + \frac{1}{2}\frac{\partial^2 V}{\partial h^2}k(t,h,c)^2 \right\} = 0 \tag{1.13}$$

1.3.2 Infinite Horizon

In this section we will derive the stochastic version of the *Hamilton Jacobi Bellman equation* for the infinite horizon with standard discounting. We can simplify equation (1.13) by eliminating explicit time dependence. We have:

$$V(t,h) = \sup_{c_t} \mathbb{E}_t \left\{ \int_t^\infty e^{-\rho\tau} u(c_\tau) d\tau \right\} \qquad \rho > 0 \tag{1.14}$$

such that $dH_t = j(t, H_t, c_t)dt + k(t, H_t, c_t)dW_t$

If we set $z = \tau - t$ then we have $dz = d\tau$, hence the above takes the form:

$$\begin{aligned} V(t,h) &= \sup_{c_t} \mathbb{E}_0 \left[\int_0^\infty e^{-\rho(z+t)} u(c_{z+t}) dz \right] \\ &= e^{-\rho t} \sup_c \mathbb{E} \left[\int_0^\infty e^{-\rho z} u(c_z) dz \;\middle|\; H_0 = h \right] \\ &= e^{-\rho t} Y(h) \end{aligned} \tag{1.15}$$

where Y(h) is as follows[3]:

$$Y(h) = \sup_c \mathbb{E} \left[\int_0^\infty e^{-\rho z} u(c_z) dz \;\middle|\; H_0 = h \right] \tag{1.16}$$

which means

$$\frac{\partial V}{\partial t} = -\rho e^{-\rho t} Y(h). \tag{1.17}$$

This essentially demonstrates the fact that in the infinite horizon setting time does not come into the optimal solution. Dropping all t dependence equation (1.13) becomes:

$$\sup_c \left\{ u(c) - \rho Y(h) + \frac{\partial Y}{\partial h} j(0,h,c) + \frac{1}{2} \frac{\partial^2 Y}{\partial h^2} k(0,h,c)^2 \right\} = 0 \tag{1.18}$$

This is known as the *Hamilton Jacobi Bellman equation* for the infinite time case.

[3]Y is a function of all state variables. In our case we only have one state variable h.

Chapter 2

Finite Horizon

The finite horizon case assumes that the investor has a finite lifespan. In other words the investor was born at time 0 and dies at time T. Now given that H_T is the terminal wealth, from [2] the investor must decide how to invest his wealth in risky and sure assets in order to maximise expected utility of wealth at the final time T. The first part of the calculation to derive the dynamics for the wealth process will follow the technique of Fouque,Papanicolaou & Sircar [8]. The derivation of the *Hamilton Jacobi Bellman equation* will follow the method outlined in Section 1.3.1. In this chapter for simplicity, we will not consider consumption.

2.1 Merton's Problem

Let us find the solution of Merton's Problem when the asset price follows the dynamics introduced by Hobson & Rogers. Our initial aim is to derive an expression for dP_t, we first observe:

$$Z_t = \log(P_t e^{-rt}) \Leftrightarrow P_t = e^{Z_t + rt} \tag{2.1}$$

An application of Itô's lemma gives us:

$$dP_t = rP_t dt + P_t dZ_t + \frac{1}{2} P_t d\langle Z \rangle_t \tag{2.2}$$

This in conjunction with equation (1.5) gives:

$$dP_t = P_t[r + \mu(S_t) + \frac{1}{2}\sigma(S_t)^2]dt + P_t[\sigma(S_t)]dW_t \tag{2.3}$$

Now an investor has wealth H_t at time t made up of α_t stocks and β_t bonds.

$$H_t = \alpha_t P_t + \beta_t e^{rt} \tag{2.4}$$

Given that the investor trades in a self financing manner, we have that the change in the value of the position over the small time interval $[t, t + dt]$ is given by:

$$dH_t = \alpha_t dP_t + \beta_t d(e^{rt}) \tag{2.5}$$

Denoting the fraction of wealth in stocks and bonds by φ_t and $(1 - \varphi_t)$ at time t respectively, we have:

$$\alpha_t = \frac{\varphi_t H_t}{P_t} \qquad\qquad \beta_t = \frac{(1-\varphi_t)H_t}{e^{rt}} \tag{2.6}$$

Thus equation (2.5) becomes:

$$dH_t = r(1 - \varphi_t)H_t dt + \frac{\varphi_t H_t}{P_t} dP_t \tag{2.7}$$

Now substituting (2.3) into (2.7) we get:

$$dH_t = [rH_t + \varphi_t H_t \mu(S_t) + \frac{1}{2}\varphi_t H_t \sigma(S_t)^2]dt + \varphi_t H_t \sigma(S_t) dW_t \tag{2.8}$$

where $H_0 = h$ and $S_0 = s$.

Remark 2.1.1. *As P_t does not appear explicitly we have a closed equation. This automatically implies that H_t is a Markov process if the control φ_t is chosen through a Markov control policy. That is if φ_t is a function of (t, H_t) [8].*

The objective of the investor is to choose φ_t such that they maximise the expected utility at some terminal time T. This is also clear from equation (1.8), as there is no consumption, the *utility of intermediate consumption* term in this equation vanishes. This leaves us with the value function:

$$V(t, h, s) = \sup_{\varphi_t} \mathbb{E}_0[U(H_T)] \tag{2.9}$$

We restrict ourselves to the case of the Power Utility function.

$$U(x) = \frac{1}{\gamma}x^\gamma, \qquad\qquad \gamma \in (0, 1) \tag{2.10}$$

where γ is known as the risk-aversion constant.[1] Hence the value function becomes:

$$V(t, h, s) = \sup_{\varphi_t} \mathbb{E}\left[\frac{H_T^\gamma}{\gamma} \,\middle|\, H_0 = h, S_0 = s\right] = \sup_{\varphi_t} \mathbb{E}^{h,s}\left[\frac{H_T^\gamma}{\gamma}\right] \tag{2.11}$$

[1]We only consider positive γ in our project for simplicity.

Theorem 2.1.1. *Given the transformation* $V(t,h,s) = \frac{1}{\gamma}h^{\gamma}m(t,s)$*, the optimal value of* φ *which satisfies equation (2.11) is:*

$$\varphi^* = \frac{1}{1-\gamma}\left[\frac{\mu(s)}{\sigma(s)^2} + \frac{1}{2} + \frac{1}{m}\frac{\partial m}{\partial s}\right] \tag{2.12}$$

Proof. We follow the strategy outlined in Section 1.3.1 to derive the *Hamilton Jacobi Bellman equation*. An application of Itô's lemma gives us:

$$\begin{aligned} dV_t = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial h}dH_t + \frac{\partial V}{\partial s}dS_t + \frac{\partial^2 V}{\partial s \partial h}d\langle H, S\rangle_t \\ + \frac{1}{2}\frac{\partial^2 V}{\partial h^2}d\langle H\rangle_t + \frac{1}{2}\frac{\partial^2 V}{\partial s^2}d\langle S\rangle_t \end{aligned} \tag{2.13}$$

This in conjunction with (1.7) and (2.8) gives:

$$\begin{aligned} dV_t = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial h}\big[rh + \varphi h\mu(s) + \frac{1}{2}\varphi h\sigma(s)^2\big]dt + \frac{\partial V}{\partial h}\varphi h\sigma(s)dW_t \\ + \frac{\partial V}{\partial s}\{[\mu(s) - \lambda s]dt + \sigma(s)dW_t\} + \frac{\partial^2 V}{\partial s \partial h}\varphi h\sigma(s)^2 dt \\ + \frac{1}{2}\frac{\partial^2 V}{\partial h^2}\varphi^2 h^2\sigma(s)^2 dt + \frac{1}{2}\frac{\partial^2 V}{\partial s^2}\sigma(s)^2 dt \end{aligned} \tag{2.14}$$

Simplifying[2],

$$\begin{aligned} V(t+dt, H_{t+dt}, S_{t+dt}) = V(t, H_t, S_t) + \Bigg[\frac{\partial V}{\partial t} + \frac{\partial V}{\partial h}\big[rh + \varphi h\mu(s) + \frac{1}{2}\varphi h\sigma(s)^2\big] \\ + \frac{\partial V}{\partial s}[\mu(s) - \lambda s] + \frac{\partial^2 V}{\partial s \partial h}\varphi h\sigma(s)^2 + \frac{1}{2}\frac{\partial^2 V}{\partial h^2}\varphi^2 h^2\sigma(s)^2 \\ + \frac{1}{2}\frac{\partial^2 V}{\partial s^2}\sigma(s)^2\Bigg]dt + \big[\frac{\partial V}{\partial h}\varphi h\sigma(s) + \frac{\partial V}{\partial s}\sigma(s)\big]dW_t \end{aligned} \tag{2.15}$$

Taking expectation at time $t = 0$, it follows that:

$$\begin{aligned} \mathbb{E}^{h,s}\Big[V(t+dt, H_{t+dt}, S_{t+dt})\Big] = V(t,h,s) + \Bigg[\frac{\partial V}{\partial t} + \frac{\partial V}{\partial h}\big[rh + \varphi h\mu(s) \\ + \frac{1}{2}\varphi h\sigma(s)^2\big] + \frac{\partial V}{\partial s}[\mu(s) - \lambda s] + \frac{\partial^2 V}{\partial s \partial h}\varphi h\sigma(s)^2 \\ + \frac{1}{2}\frac{\partial^2 V}{\partial h^2}\varphi^2 h^2\sigma(s)^2 + \frac{1}{2}\frac{\partial^2 V}{\partial s^2}\sigma(s)^2\Bigg]dt \end{aligned} \tag{2.16}$$

[2]Note that $dV_t = V(t+dt, H_{t+dt}, S_{t+dt}) - V(t, H_t, S_t)$

In our case, the Bellman principle of optimality given in equation (1.9) can be expressed as:

$$\sup_{\varphi_t}\Big\{-V(t,h,s)+\mathbb{E}^{h,s}\Big[V(t+dt,H_{t+dt},S_{t+dt})\Big]\Big\}=0 \tag{2.17}$$

at optimum φ_t. Hence it follows that our *Hamilton Jacobi Bellman equation* is:

$$\begin{aligned}\sup_{\varphi}\Big\{\frac{\partial V}{\partial t}+\frac{\partial V}{\partial h}\big[rh+\varphi h\mu(s)+\frac{1}{2}\varphi h\sigma(s)^2\big]+\frac{\partial V}{\partial s}[\mu(s)-\lambda s]\\ +\frac{\partial^2 V}{\partial s\partial h}\varphi h\sigma(s)^2+\frac{1}{2}\frac{\partial^2 V}{\partial h^2}\varphi^2h^2\sigma(s)^2+\frac{1}{2}\frac{\partial^2 V}{\partial s^2}\sigma(s)^2\Big\}=0\end{aligned} \tag{2.18}$$

To simplify our PDE further we can apply some transformations that eliminate the $\frac{\partial V}{\partial h}$, $\frac{\partial^2 V}{\partial s\partial h}$ and $\frac{\partial^2 V}{\partial h^2}$ terms.[3] The reason is to reduce the PDE to a state in which it consists of derivatives only in one variable. Consider the transformation:

$$V(t,h,s)=\frac{1}{\gamma}h^{\gamma}m(t,s) \tag{2.19}$$

This means equation (2.18) becomes:

$$\begin{aligned}\frac{\partial m}{\partial t}+mr\gamma+\frac{\partial m}{\partial s}[\mu(s)-\lambda s]+\frac{1}{2}\frac{\partial^2 m}{\partial s^2}\sigma(s)^2+\sup_{\varphi}\Big\{\varphi\big[\mu(s)m\gamma+\frac{1}{2}m\sigma(s)^2\gamma\\ +\frac{\partial m}{\partial s}\gamma\sigma(s)^2\big]+\frac{1}{2}\varphi^2\gamma(\gamma-1)m\sigma(s)^2\Big\}=0\end{aligned} \tag{2.20}$$

with boundary condition:

$$m(T,s)=1. \tag{2.21}$$

By taking derivatives of equation (2.20) with respect to φ and equating it to 0, we get the supremum:

$$\varphi^*=\frac{1}{1-\gamma}\left[\frac{\mu(s)}{\sigma(s)^2}+\frac{1}{2}+\frac{1}{m}\frac{\partial m}{\partial s}\right] \tag{2.22}$$

This is the supremum because in equation (2.18) we are maximising a quadratic in φ, now as $\gamma\in(0,1)$ the quadratic coefficient is negative. Thus this is the optimal φ. □

[3]One could choose a transformation to eliminate $\frac{\partial V}{\partial s}$, $\frac{\partial^2 V}{\partial s\partial h}$ and $\frac{\partial^2 V}{\partial s^2}$ instead of what was eliminated above. However, this would be unnecessarily tedious.

Corollary 2.1.1. *Given the transformation $m(t,s) = n(t,s)^{1-\gamma}$, the solution of the Hamilton Jacobi Bellman equation (2.20) can be written as the expectation:*

$$n(t,s) = \mathbb{E}^{t,s}\left[e^{\int_t^T \Gamma(\tilde{S}_\zeta)d\zeta}\right] \tag{2.23}$$

where

$$\Gamma(s) = \frac{(2\mu(s)+\sigma(s)^2)^2\gamma}{8(1-\gamma)^2\sigma(s)^2} + \frac{r\gamma}{1-\gamma} \tag{2.24}$$

and $\tilde{S}$ satisfies:

$$d\tilde{S}_t = \left[\frac{\gamma(2\mu(\tilde{S}_t)+\sigma(\tilde{S}_t)^2)}{2(1-\gamma)} + \mu(\tilde{S}_t) - \lambda\tilde{S}_t\right]dt + \sigma(\tilde{S}_t)dW_t \tag{2.25}$$

with $\tilde{S}_t = s$.

Proof. Insert φ^* into (2.20) and get:

$$\frac{\partial m}{\partial t} + \left[\frac{(2\mu(s)+\sigma(s)^2)^2\gamma}{8(1-\gamma)\sigma(s)^2} + r\gamma\right]m + \left[\frac{\gamma(2\mu(s)+\sigma(s)^2)}{2(1-\gamma)} + \mu(s) - \lambda s\right]\frac{\partial m}{\partial s} + \frac{\gamma\sigma(s)^2}{2(1-\gamma)}\frac{1}{m}\left(\frac{\partial m}{\partial s}\right)^2 + \frac{1}{2}\sigma(s)^2\frac{\partial^2 m}{\partial s^2} = 0 \tag{2.26}$$

In view of the paper by Musiela & Zariphopoulou [9], we can linearise equation (2.26) by applying the following transformation:

$$m(t,s) = n(t,s)^{1-\gamma} \tag{2.27}$$

Hence equation (2.26) becomes:

$$\frac{\partial n}{\partial t} + \left[\frac{(2\mu(s)+\sigma(s)^2)^2\gamma}{8(1-\gamma)^2\sigma(s)^2} + \frac{r\gamma}{1-\gamma}\right]n + \left[\frac{\gamma(2\mu(s)+\sigma(s)^2)}{2(1-\gamma)} + \mu(s) - \lambda s\right]\frac{\partial n}{\partial s} + \frac{1}{2}\sigma(s)^2\frac{\partial^2 n}{\partial s^2} = 0 \tag{2.28}$$

Applying the Feynman-Kac formula to equation (2.26) we obtain the required expectation. □

All that remains is to numerically approximate this expectation. In Chapter 4, we will use Monte Carlo methods to do this.

Chapter 3

Infinite Horizon

Let us consider the infinite horizon case. In this scenario there is no upper bound T on time. This is very useful for investors that have long-term investment goals, such as a pension fund or an insurance company. In such cases, an infinite horizon is more appropriate as a natural fixed finite horizon does not exist. One can view an infinite horizon as an approximation to a 'long' horizon. This observation forms the basis for this chapter. We will first look at a finite horizon problem with consumption. Then through implementing the so-called *transversality condition* [1],[7] & [32], we can take $T \to \infty$.

3.1 Merton's Problem

Before we begin, let us refer back to equation (1.8). As there is no concept of terminal time for the infinite horizon case, the *utility of terminal wealth* is nonexistent. Thus, we need to introduce consumption. Our wealth process (2.8) now becomes:

$$dH_t = [rH_t + \varphi_t H_t \mu(S_t) + \frac{1}{2}\varphi_t H_t \sigma(S_t)^2 - c_t]dt + \varphi_t H_t \sigma(S_t) dW_t \qquad (3.1)$$

where $H_0 = h$, $S_0 = s$ and c_t denotes consumption. Now the objective for the investor is to choose a φ_t and a consumption c_t such that they maximise the expected utility. Again, from equation (1.8) our value function can be defined as:

$$V(t,h,s) = \sup_{\varphi_t, c_t} \mathbb{E}_t \left[\int_t^T e^{-\rho\tau} u(c_\tau) d\tau \right] \qquad \rho > 0 \qquad (3.2)$$

We restrict ourselves to the case of the Power Utility function.

$$u(c_\tau) = \frac{1}{\gamma} c_\tau^\gamma, \qquad \gamma \in (0,1) \tag{3.3}$$

where as before, γ is the risk-aversion constant.

Let us remove explicit time dependence. Let $z = \tau - t$ then $dz = d\tau$, hence following Section 1.3.2 we have:

$$V(t,h,s) = e^{-\rho t} Y(s,h) \tag{3.4}$$

where $Y(s,h)$ is:

$$Y(s,h) = \sup_{\varphi,c} \mathbb{E}^{h,s}\left\{ \int_0^\infty e^{-\rho z} u(c_z) dz \right\} \tag{3.5}$$

Now taking into account that our wealth process (3.1) now includes consumption, equation (2.18) modifies into:

$$\sup_{\varphi,c}\left\{ u(c) + \frac{\partial V}{\partial t} + \frac{\partial V}{\partial h}\left[rh + \varphi h \mu(s) + \frac{1}{2}\varphi h \sigma(s)^2 - c\right] + \frac{\partial V}{\partial s}[\mu(s) - \lambda s] + \frac{\partial^2 V}{\partial s \partial h}\varphi h \sigma(s)^2 + \frac{1}{2}\frac{\partial^2 V}{\partial h^2}\varphi^2 h^2 \sigma(s)^2 + \frac{1}{2}\frac{\partial^2 V}{\partial s^2}\sigma(s)^2 \right\} = 0 \tag{3.6}$$

Using the relation between V and Y as indicated in equation (3.4), substituting for the derivatives of V into equation (3.6), we get our *Hamilton Jacobi Bellman equation*:

$$\sup_{\varphi,c}\left\{ u(c) - \rho Y + \frac{\partial Y}{\partial h}\left[rh + \varphi h \mu(s) + \frac{1}{2}\varphi h \sigma(s)^2 - c\right] + \frac{\partial Y}{\partial s}[\mu(s) - \lambda s] + \frac{\partial^2 Y}{\partial s \partial h}\varphi h \sigma(s)^2 + \frac{1}{2}\frac{\partial^2 Y}{\partial h^2}\varphi^2 h^2 \sigma(s)^2 + \frac{1}{2}\frac{\partial^2 Y}{\partial s^2}\sigma(s)^2 \right\} = 0 \tag{3.7}$$

Rather than the boundary condition (2.21) we add the technical *transversality condition* [32].

$$\lim_{T\to\infty} \mathbb{E}[e^{-\int_t^T \rho dk} Y(s,h) | \mathcal{F}_t] = 0 \tag{3.8}$$

Theorem 3.1.1. *Given the transformation $Y(h,s) = h^\gamma m(s)$, the optimal value of c and φ which satisfy equation (3.7) are:*

1.

$$c^* = \left(\frac{\partial Y}{\partial h}\right)^{-\frac{1}{1-\gamma}} \tag{3.9}$$

2.

$$\varphi^* = \frac{1}{1-\gamma}\left[\frac{\mu(s)}{\sigma(s)^2} + \frac{1}{2} + \frac{1}{m}\frac{\partial m}{\partial s}\right] \tag{3.10}$$

Proof. By differentiating equation (3.7) with respect to c, it is straightforward to see that:

$$-\frac{\partial Y}{\partial h} + \frac{\partial u}{\partial c} = 0 \Rightarrow c = \left(\frac{\partial Y}{\partial h}\right)^{-\frac{1}{1-\gamma}} \tag{3.11}$$

Taking second derivatives of equation (3.7) with respect to c, we get:

$$\frac{\partial^2 u}{\partial c^2} = (\gamma - 1)c^{\gamma-2} \tag{3.12}$$

which is negative for all $\gamma \in (0, 1)$. Hence, the optimal value of c is obtained when:

$$c^* = \left(\frac{\partial Y}{\partial h}\right)^{-\frac{1}{1-\gamma}} \tag{3.13}$$

This proves first part of the theorem. Inserting c^* into equation (3.7) we get:

$$-\rho Y + \frac{\partial Y}{\partial h}rh + \left[\frac{1-\gamma}{\gamma}\right]\left(\frac{\partial Y}{\partial h}\right)^{-\frac{\gamma}{1-\gamma}} + \frac{\partial Y}{\partial s}[\mu(s) - \lambda s] + \frac{1}{2}\frac{\partial^2 Y}{\partial s^2}\sigma(s)^2$$
$$+\sup_{\varphi}\left\{\frac{\partial Y}{\partial h}\left[\varphi h\mu(s) + \frac{1}{2}\varphi h\sigma(s)^2\right] + \frac{\partial^2 Y}{\partial s\partial h}\varphi h\sigma(s)^2 + \frac{1}{2}\frac{\partial^2 Y}{\partial h^2}\varphi^2 h^2\sigma(s)^2\right\} = 0 \tag{3.14}$$

Now we must apply a transformation to reduce the PDE into a state that consists of only single variable derivatives. Consider the transformation:

$$Y(h, s) = h^\gamma m(s) \tag{3.15}$$

This means equation (3.14) becomes:

$$-m\rho + mr\gamma + \left(\frac{1-\gamma}{\gamma}\right)\gamma^{-\frac{\gamma}{1-\gamma}}m^{-\frac{\gamma}{1-\gamma}} + \frac{\partial m}{\partial s}[\mu(s) - \lambda s] + \frac{1}{2}\frac{\partial^2 m}{\partial s^2}\sigma(s)^2$$
$$+\sup_{\varphi}\left\{\varphi\left[\mu(s)m\gamma + \frac{1}{2}m\sigma(s)^2\gamma + \frac{\partial m}{\partial s}\gamma\sigma(s)^2\right] + \frac{1}{2}\varphi^2\gamma(\gamma-1)m\sigma(s)^2\right\} = 0 \tag{3.16}$$

Finally the supremum over φ is attained at:

$$\varphi^* = \frac{1}{1-\gamma}\left[\frac{\mu(s)}{\sigma(s)^2} + \frac{1}{2} + \frac{1}{m}\frac{\partial m}{\partial s}\right] \tag{3.17}$$

As mentioned in Chapter 2, this is the supremum because in equation (3.16) we are maximising a quadratic in φ, now as $\gamma \in (0,1)$ the quadratic coefficient is negative. Thus this is the optimal φ. □

Notice that the optimal φ^* is identical to that of the finite horizon. This is what we expect, as our consumption level c is independent of the fraction of wealth in the risky asset φ.

Corollary 3.1.1. *Given the transformation $m(s) = n(s)^{1-\gamma}$, the solution of the Hamilton Jacobi Bellman equation (3.7) can be written as the expectation:*

$$n(t,s) = \mathbb{E}^{t,s}\left[\int_t^T e^{\int_t^b \Psi(\tilde{S}_\zeta)d\zeta}\gamma^{-\frac{1}{1-\gamma}}db + e^{\int_t^T \Psi(\tilde{S}_\zeta)d\zeta}\right] \tag{3.18}$$

where

$$\Psi(s) = \frac{(2\mu(s)+\sigma(s)^2)^2\gamma}{8(1-\gamma)^2\sigma(s)^2} + \frac{r\gamma-\rho}{\gamma} \tag{3.19}$$

and $\tilde{S}$ satisfies:

$$d\tilde{S}_t = \left[\frac{\gamma(2\mu(\tilde{S}_t)+\sigma(\tilde{S}_t)^2)}{2(1-\gamma)} + \mu(\tilde{S}_t) - \lambda\tilde{S}_t\right]dt + \sigma(\tilde{S}_t)dW_t \tag{3.20}$$

with $\tilde{S}_t = s$ and $n(T,s) = 1$.

Proof. Now we insert φ^* into (3.16) and get:

$$\left[\frac{(2\mu(s)+\sigma(s)^2)^2\gamma}{8(1-\gamma)\sigma(s)^2} + r\gamma - \rho\right]m + \left[\frac{\gamma(2\mu(s)+\sigma(s)^2)}{2(1-\gamma)} + \mu(s) - \lambda s\right]\frac{\partial m}{\partial s} + \frac{\gamma\sigma(s)^2}{2(1-\gamma)}\frac{1}{m}\left(\frac{\partial m}{\partial s}\right)^2 + \frac{1}{2}\sigma(s)^2\frac{\partial^2 m}{\partial s^2} + \left(\frac{1-\gamma}{\gamma}\right)\gamma^{-\frac{\gamma}{1-\gamma}}m^{-\frac{\gamma}{1-\gamma}} = 0 \tag{3.21}$$

Like the previous chapter, we can linearise equation (3.21) by applying the following transformation:

$$m(s) = n(s)^{1-\gamma} \tag{3.22}$$

Hence, equation (3.21) becomes:

$$\left[\frac{(2\mu(s)+\sigma(s)^2)^2\gamma}{8(1-\gamma)^2\sigma(s)^2}+\frac{r\gamma-\rho}{\gamma}\right]n+\left[\frac{\gamma(2\mu(s)+\sigma(s)^2)}{2(1-\gamma)}+\mu(s)-\lambda s\right]\frac{\partial n}{\partial s} \\ +\frac{1}{2}\sigma(s)^2\frac{\partial^2 n}{\partial s^2}+\gamma^{-\frac{1}{1-\gamma}}=0 \tag{3.23}$$

Using the boundary condition $n(T,s)=1$, we apply the Feynman-Kac formula to equation (2.26) and obtain the required expectation. □

Corollary 3.1.2. *Equation (3.18) can be written as the expectation:*

$$n(s)=\mathbb{E}^{t,s}\left[\int_t^\infty e^{\int_t^b \Psi(\tilde{S}_\zeta)d\zeta}\gamma^{-\frac{1}{1-\gamma}}db\right] \tag{3.24}$$

Proof. Consider equation (3.18), taking $T\to\infty$ and applying the *transversality condition* proves the result. □

Chapter 4

Numerical Analysis Finite Horizon

In this chapter, using Microsoft Excel we aim to solve numerically the finite horizon PDE derived earlier. We have already applied the Feynman-Kac formula and reduced the PDE into an expectation; all that remains is to run Monte Carlo simulations to solve the PDE. We will compare the parameters in the Hobson & Rogers model with that of classical short rate models. We now need to choose suitable functions for $\sigma(s)$ and $\mu(s)$ bearing in mind that they have to be Lipschitz functions, the former being positive as well.

Let us consider basing the parameters on the Dothan model [24] & [25], which has dynamics:

$$d\tilde{S}_t = \lambda \tilde{S}_t dt + \xi \tilde{S}_t dW_t \tag{4.1}$$

where λ and ξ are positive constants.[1] The reason we choose this model is because the $\tilde{S}_t$ is lognormally distributed, implying that $\tilde{S}_t$ is always strictly positive for each t [25]. This further highlights the models admissibility in our framework, as $\xi\tilde{S}_t$ will also be strictly positive. Thus taking:

$$\sigma(s) = \xi s \wedge N_1 \tag{4.2}$$

for some large constant N_1.[2] We satisfy both the strict positivity and Lipschitz conditions on $\sigma(s)$. This choice also captures the phenomenon that if the current price significantly differs from the past average, then the volatility

[1] The Dothan model actually allows $\lambda \in \mathbb{R}$, however from the Hobson & Rogers model, λ is a positive constant that expresses the discounting rate of past information.

[2] The use of the technical condition N_1 is reminiscent of Hobson & Rogers [5]. They use this to ensure that the function is bounded.

is high. Let us now find a function for $\mu(s)$, consider:

$$\mu(s) = [2\lambda(1-\gamma)s - \frac{1}{2}\xi^2\gamma s^2] \wedge N_2 \tag{4.3}$$

for some small negative constant N_2. This function is Lipschitz as it is bounded. Thus with these choices of $\sigma(s)$ and $\mu(s)$ our equations (2.12) and (2.25) become:

$$\varphi^* = \frac{1}{2} + \frac{2\lambda}{\xi^2 s} + \frac{1}{n}\frac{\partial n}{\partial s} \tag{4.4}$$

$$\Gamma(s) = \frac{(4\lambda + \xi^2 s)^2\gamma}{8\xi^2} + \frac{r\gamma}{1-\gamma} \tag{4.5}$$

with $\tilde{S}_t = s$. Moreover in equation (4.4), we used the transformation (2.27). We have simplified our SDE (2.25) to the form required for the Dothan model. In order for us to simulate this, we will need to be able to generate Brownian motion. We can do this by using the method of Linear Congruential Generators to simulate uniform random variables. Then using the Box-Muller algorithm we can obtain standard normal random variables. Now recall from Brigo & Mercurio [25] that the dynamics of equation (4.1) can be integrated as follows:

$$\tilde{S}_T = \tilde{S}_t e^{(\lambda - \frac{1}{2}\xi^2)(T-t) + \xi(W_T - W_t)} \tag{4.6}$$

where $t < T$. Furthermore $W_T - W_t \sim \mathcal{N}(0, T-t)$.

Referring to Glasserman [29], we see that in order to simulate the standard Brownian motions $(W_{t_1}, \cdots, W_{t_n})$ at a fixed set of points $0 < t_1 < \cdots < t_n$; we will need to consider the set $(Z_1, \cdots, Z_n)$ of independent standard normal random variables. We set $t_0 = 0$ and $W_{t_0} = 0$ and generate as follows:

$$W_{t_{i+1}} - W_{t_i} = \sqrt{t_{i+1} - t_i} Z_{i+1} \qquad i = 0, \cdots, n-1 \tag{4.7}$$

In our case, we will divide time into equally spaced discrete intervals each with width $\frac{1}{1000}$. In other words if we discretise time into n steps, then:

$$t_{i+1} - t_i = \frac{1}{1000} \qquad \forall i = 0, \cdots, n-1 \tag{4.8}$$

Thus by taking $\tilde{S}_{t_0} = 1$, we can simulate our offset function as follows:

$$\tilde{S}_{ti+1} = \tilde{S}_{ti} e^{\frac{1}{1000}(\lambda - \frac{1}{2}\xi^2) + \frac{\xi}{\sqrt{1000}} Z_{i+1}} \tag{4.9}$$

In the next two sections, I will introduce the methods we will use to generate uniform and standard normal random variables.

4.1 Linear Congruential Generators

We require a method of creating uniform random values in the set $(0, 1)$. In this section, we introduce the simplest class of generators of random numbers, the Linear Congruential Generators. This generation method essentially produces a series of integers $x_0, x_1, x_2, \cdots$ by the following calculation:

$$x_{n+1} = (ax_n + b) mod(c) \qquad a, b, c \in \mathbb{N}^* \tag{4.10}$$

All that remains is to choose an initial value, sometimes called the seed x_0.

Pseudo random numbers produced by a Linear Congruential Generator behave somewhat like uniform random values in the set $(0, \cdots, c-1)$, but there are problems [27]. By the *pigeonhole principle*[3], the sequence $x_0, x_1, x_2, \cdots$ must repeat itself over and over. As soon as the first period ends, we are able to predict the second period, negating the point of a random number generator. One way to solve this is if we take c sufficiently large. The period is at most c and in the majority of cases less than that. To ensure we get a maximised period, we must choose our constants a, b, c so that the following conditions are met:

1. b and c are relatively prime.
2. $a - 1$ is divisible by all prime factors of c.
3. $a - 1$ is a multiple of 4 if c is a multiple of 4.

Park & Miller [28] suggested the 'Minimal Standard' parameters: $a = 16807$, $b = 0$ and $c = 2147483647$. We will also use these parameters in our analysis.

4.2 Box-Muller Algorithm

In the previous section, we examined an algorithm that generates uniform random values. Here, we discuss the Box-Muller algorithm [26], which generates normal random values. This straightforward method begins with generating two uniform random variables $U_1, U_2 \sim U(0, 1)$. Then we consider

[3]The pigeonhole principle frequently arises in computer science. The basic idea is that if we have a pigeonholes and b pigeons and $a < b$ where $a, b \in \mathbb{N}$. Then if we put each pigeon in a hole there must be at least one pigeonhole with two pigeons.

the random variables:

$$X_1 = \sqrt{-2 \log U_1} \cos 2\pi U_2 \tag{4.11}$$

$$X_2 = \sqrt{-2 \log U_1} \sin 2\pi U_2 \tag{4.12}$$

(X_1, X_2) will be a pair of independent normal random variables with mean zero and unit variance.

4.3 Numerical Integration

Now recall that our aim in this chapter is to numerically approximate the solution to the right hand side of equation (2.23). Having already established a procedure for deriving $\tilde{S}_t$, it is straightforward given equation (4.5) to extend this method so we can generate $\Gamma(s)$. However just being able to simulate $\Gamma(s)$ is not enough, equation (2.23) tells us that we need to evaluate the integral:

$$\int_{t_i}^{t_{i+1}} \Gamma(\tilde{S}_\zeta) d\zeta \tag{4.13}$$

We will have to evaluate this numerically. The idea behind our method would be to divide the area under the graph into tiny rectangular strips as shown in Figure (4.1). Using elementary geometry, we will then individually work out the area of each strip and then sum them. This procedure is often called *Riemann sums*, in particular we will adopt the method of *Right Riemann sums* in our analysis, with a strip length of $\frac{1}{1000}$. It is also worth mentioning that due to the nowhere differentiability property of the sample path for $\Gamma(\tilde{S}_t)$, we will never be able to exactly work out this area. We can only improve our approximations by dividing the area into thinner and thinner strips. Finally taking exponentials, we can numerically estimate the quantity:

$$e^{\int_{t_i}^{t_{i+1}} \Gamma(\tilde{S}_\zeta) d\zeta} \tag{4.14}$$

4.4 Numerical Differentiation

Observe that in equation (4.4), we have to work out the derivative $\frac{\partial n}{\partial s}$. We can see this will be difficult as s is simulated from a stochastic function. An

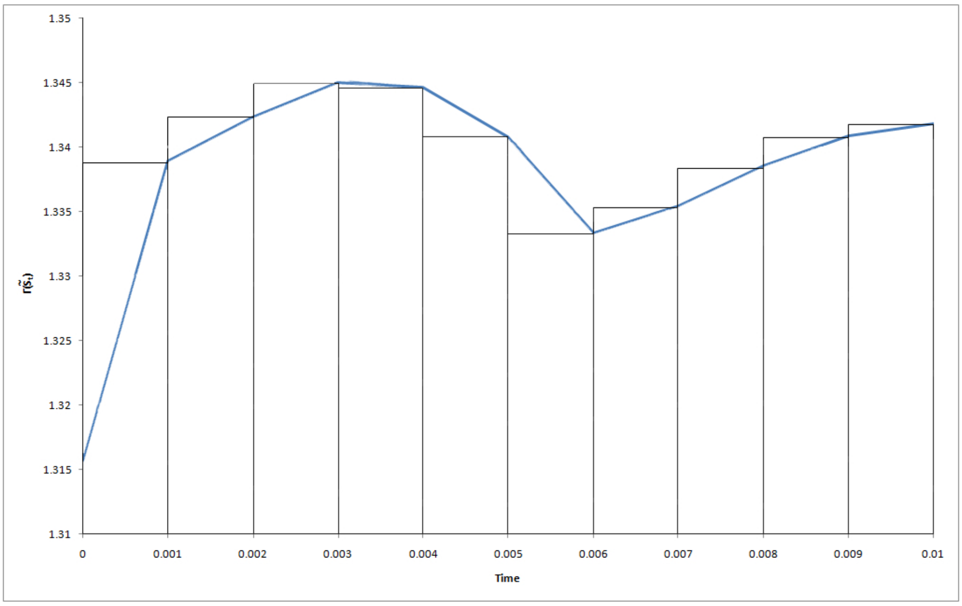

Figure 4.1: A plot to demonstrate the idea of numerical integration. In this example we took the strip width to be $\frac{1}{1000}$

analytic derivative will never exist for $n(t, s)$. We will take the approach by Benth [31] to numerically solve this.

$$\frac{\partial n(t,s)}{\partial s} = \lim_{\Delta \to 0} \frac{n(t, s+\Delta) - n(t,s)}{\Delta} \tag{4.15}$$

We can numerically approximate this with:

$$\frac{\partial n(t,s)}{\partial s} \approx \frac{n(t, s+\Delta) - n(t,s)}{\Delta} \tag{4.16}$$

for small Δ.

4.5 Monte Carlo Simulations

What we need to do now in order to approximate the expectation defined in equation (2.23) is to repeat the process of approximating $e^{\int_{t_i}^{t_{i+1}} \Gamma(\tilde{S}_\zeta) d\zeta}$ for many different time intervals, then averaging them. In other words each $\frac{1}{1000}$ unit we progress in time we evaluate another area strip under $\Gamma(s)$, exponentiate this and then average it with the previous value of the simulation. In

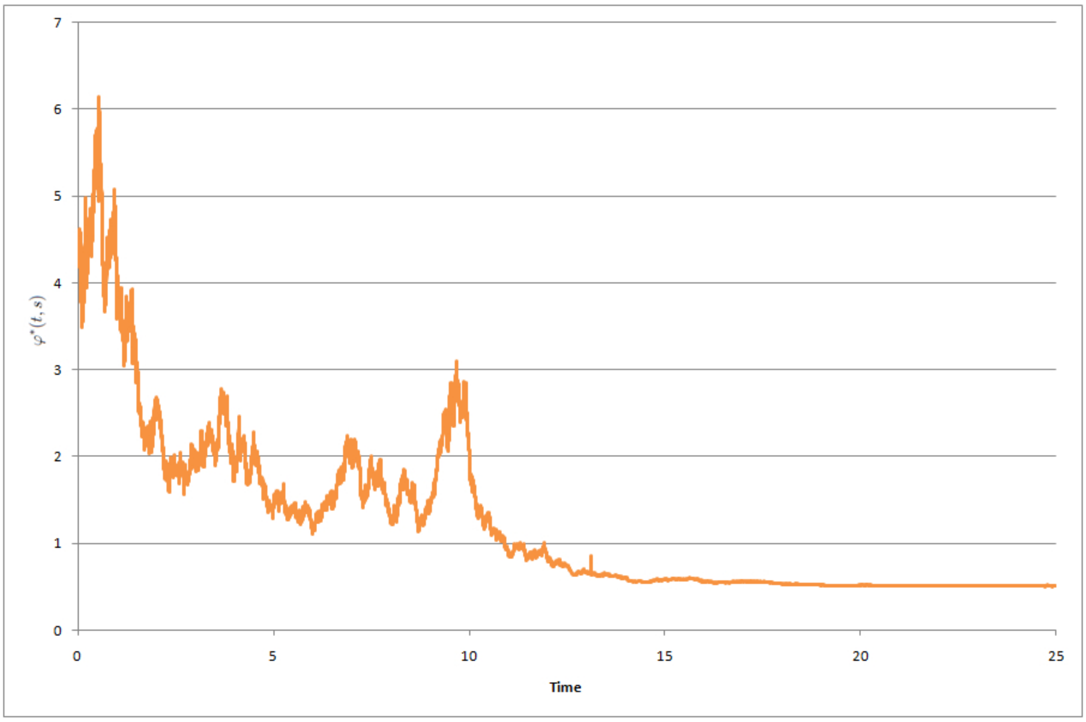

Figure 4.2: A plot to demonstrate the evolution of $\varphi^*(t, s)$ over time and offsets. We performed 25000 simulations.

our case we have simulated this 25000 times by incrementing time in steps of $\frac{1}{1000}$ from 0 to 25.

We have avoided using Microsoft Excel's built in normal distribution generator, as its performance is limited [30]. Instead, we opt for a Box-Muller approach implemented on C++. I have provided the code in the appendix (A.1). Our ultimate aim is to be able to evaluate equation (4.4) and solve for φ^*. We will examine the evolution of φ^* as we vary time and offset. Figure (4.2) shows the relationship of φ^* against time and offsets. Figure (4.3) illustrates the evolution of φ^* over the offsets, while we keep time fixed. We have used parameters $\lambda = 0.5$, $\xi = 0.5$, $\gamma = 0.5$, $r = 0.05$, $N_1 = 25$ and $N_2 = -25$.

Notice that φ^* must be less than 1 as it is the fraction of the wealth in risky assets. We can see in both plots that indeed once we do enough simulations it does indeed fall to a level below 1.

In Figure (4.3), we only plotted the first 10000 simulations due to system limitations. Computer instability prevented us from plotting further points.

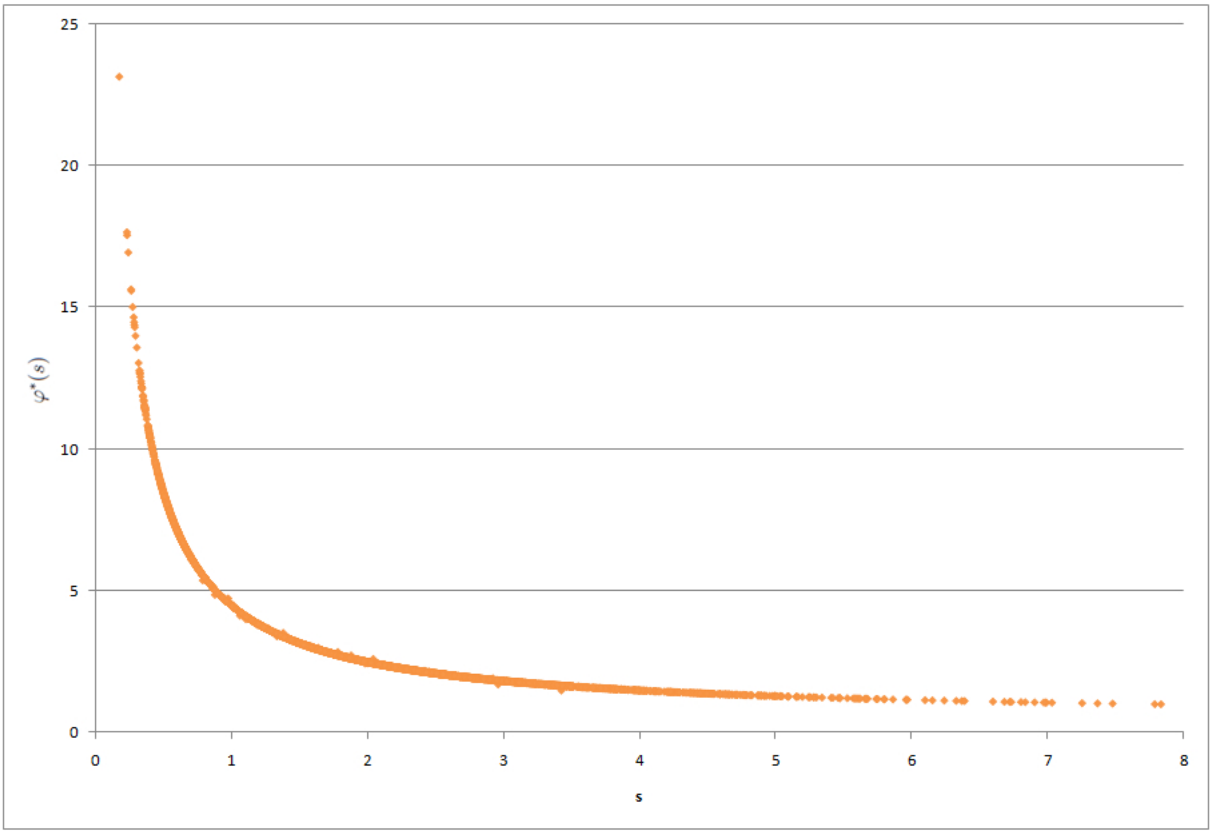

Figure 4.3: A scatter plot to demonstrate the relationship of $\varphi^*(s)$ and the offsets, while we keep time fixed. We plotted the first 10000 simulations.

As the points were very closely clustered together, it caused my computer system to consume more memory to render the image. Nevertheless, we still capture the fundamentals in both plots. We can clearly see that in Figure (4.2), as time progresses φ^* gets smaller and smaller. In fact taking limits, we get:

$$\lim_{t \to \infty} [\varphi^*] = \frac{1}{2} \tag{4.17}$$

This is what we expected, as when we go further in time the offset increases as well. The larger the offset value, the greater the difference in the time between the observations of the stock price; thus more inaccurate representations of stock dynamics. Generally the smaller the offset the more accurately you can capture the movements in the market. Therefore, it is only natural that the investor reduces their proportion of wealth in the risky asset. In Figure (4.3), we fixed $S_0 = 1$ and we froze the time interval to $t = 0$ and $T = 1$. We then generated 10000 values for the offset. As shown on the diagram as the number of offsets increase the holdings in the risky assets decrease. This further reiterates the above idea.

4.6 The Parameters

In this section, we discuss the parameters λ, ξ and γ and observe the effects varying them has on $\varphi^*(s)$.

4.6.1 The Risk Aversion Constant γ

Lets observe the effects of changing the risk-aversion constant γ. Remember that our utility function is of the power utility type as defined in equation (2.10).

4.6.2 Risk Attitude

There are three different attitudes an investor can have when it comes to risk:

- Risk Prone.
- Risk Neutral.

- Risk Averse.

The Risk Prone Investor

This mindset corresponds to when:

$$\frac{d}{dx}U(x) > 0 \Leftrightarrow \gamma > 1. \tag{4.18}$$

The Risk Neutral Investor

This mindset corresponds to when:

$$\frac{d}{dx}U(x) = 0 \Leftrightarrow \gamma = 1. \tag{4.19}$$

As a utility function of the power type requires $\gamma \in (0, 1)$, we have that this function is unsuitable to model the utility of a risk neutral or a risk prone investor.

The Risk Averse Investor

This mindset corresponds to when:

$$\frac{d}{dx}U(x) < 0 \Leftrightarrow \gamma < 1. \tag{4.20}$$

In contrast to the previous two cases, this mindset is admissible in the power utility. Generally as γ gets small the investor becomes more and more risk averse.

Now looking at equation (4.4), we notice that γ is not explicitly present. In fact the only place γ is used is during the derivation of $n(t, s)$ when we worked out $\Gamma(s)$ (4.5). Even in $\Gamma(s)$, γ plays a passive role. Once we actually average it out to derive $n(t, s)$ any effect γ had was further dampened. Therefore the effect of varying γ on $\varphi^*(s)$ will be miniscule. In our case we have plotted the relationship of $\varphi^*(s)$ and γ with the parameters $\xi = 0.5$, $\lambda = 0.001$, $r = 0.05$, $N_1 = 25$, $N_2 = -25$ and we fix time to the interval $[0, 1]$. Clearly, Figure (4.4) shows that as the investor's risk aversion constant increases so does the proportion of their wealth in the risky asset. This was expected, the investor is becoming more of a risk taker.

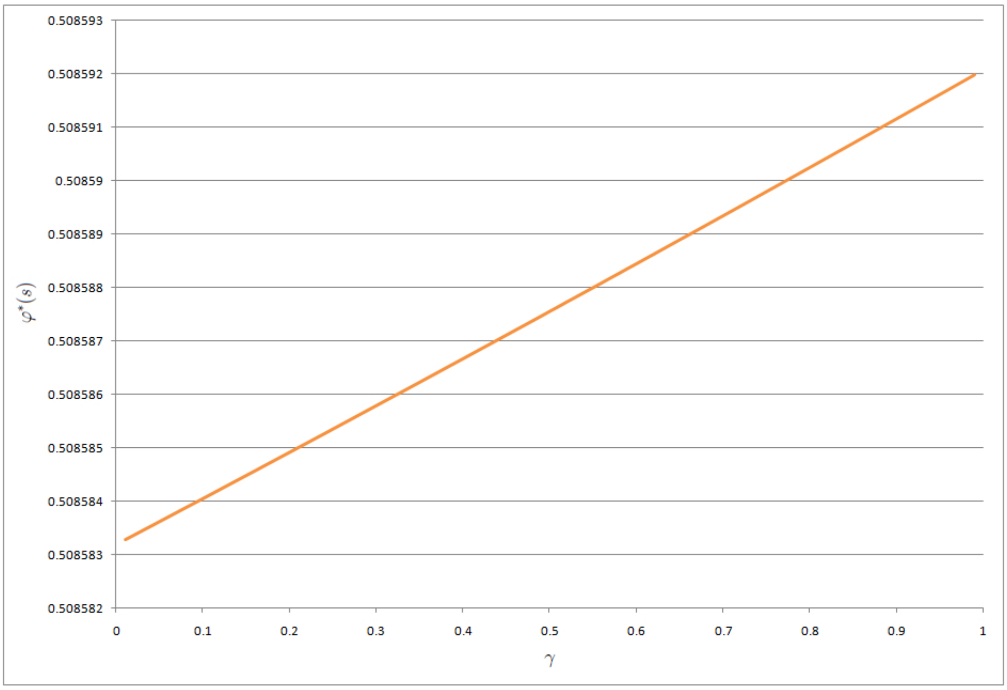

Figure 4.4: A graph demonstrating the relationship of $\varphi^*(s)$ and γ

4.6.3 The Volatility Parameter ξ

The standard deviation parameter ξ is a positive constant associated with the volatility of the process $\tilde{S}_t$. Now we must remember that $0 < \varphi < 1$. This means that we must truncate the graph so that this constraint holds. In Figure (4.5) we have plotted the relationship of $\varphi^*(s)$ and ξ with the parameters $\gamma = 0.5$, $\lambda = 0.5$, $r = 0.05$, $N_1 = 25$ and $N_2 = -25$. The $\varphi^*(s)$ we have used here has been derived using 10000 simulations as we vary ξ, we also fixed time as before.

This graph turned out to be exactly how we expected. As the volatility increased, the market became more and more unpredictable. This graph captures the natural reaction of the investor by decreasing his proportion in the risky assets. We can see clearly that from equation (4.4):

$$\lim_{\xi \to 0}[\varphi^*] = \infty \tag{4.21}$$

This means that for some ξ, regardless of the other parameters $\varphi^* > 1$. This contradicts our initial constraint on φ^*. Therefore, the only way to satisfy the initial constraint is to restrict volatility. However, the problem

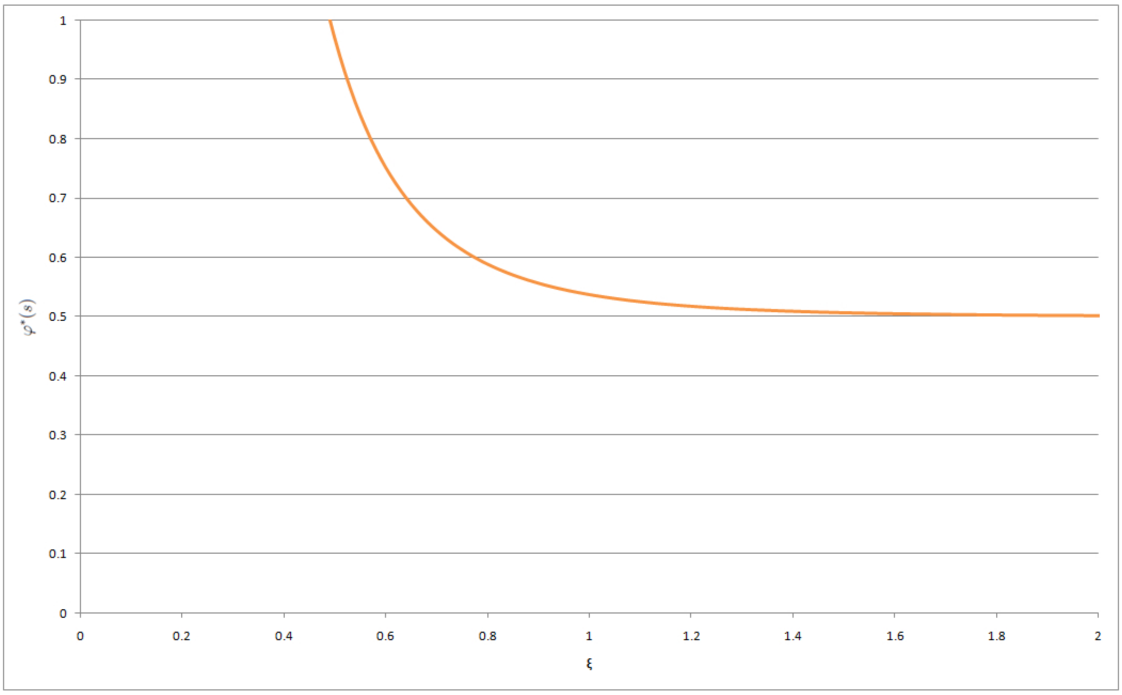

Figure 4.5: A graph demonstrating the relationship of $\varphi^*(s)$ and ξ

that lies here is that volatility is not observable, so we won't be able to tell whether the volatility of the market will allow admissible values of φ^*. We will discuss this further and examine alternatives in Chapter 6.

4.6.4 The Discounting Rate of Past Information λ

Now we will consider the effects of changing λ, the discounting rate of past information in the Hobson & Rogers model. We have shown in Figure (4.6) the relationship of $\varphi^*(s)$ and λ with the parameters $\gamma = 0.5$, $\xi = 0.51$, $r = 0.05$, $N_1 = 25$ and $N_2 = -25$. The $\varphi^*(s)$ we have used here has been derived using 10000 simulations as we vary λ, we also fixed time as before.

This graph is exactly how we expected it to be. λ in a sense depends on the past stocks that modelled the volatility. Generally, the bigger the time gap between the log prices of the stocks the more we will discount and the larger λ will be. The converse is also true. Therefore we will be able to model the volatility more realistically the less we discount, simply because we will be basing our volatility on recent stocks. Thus as Figure (4.6) shows, when λ is small the investor takes advantage of the realistic approximations of the

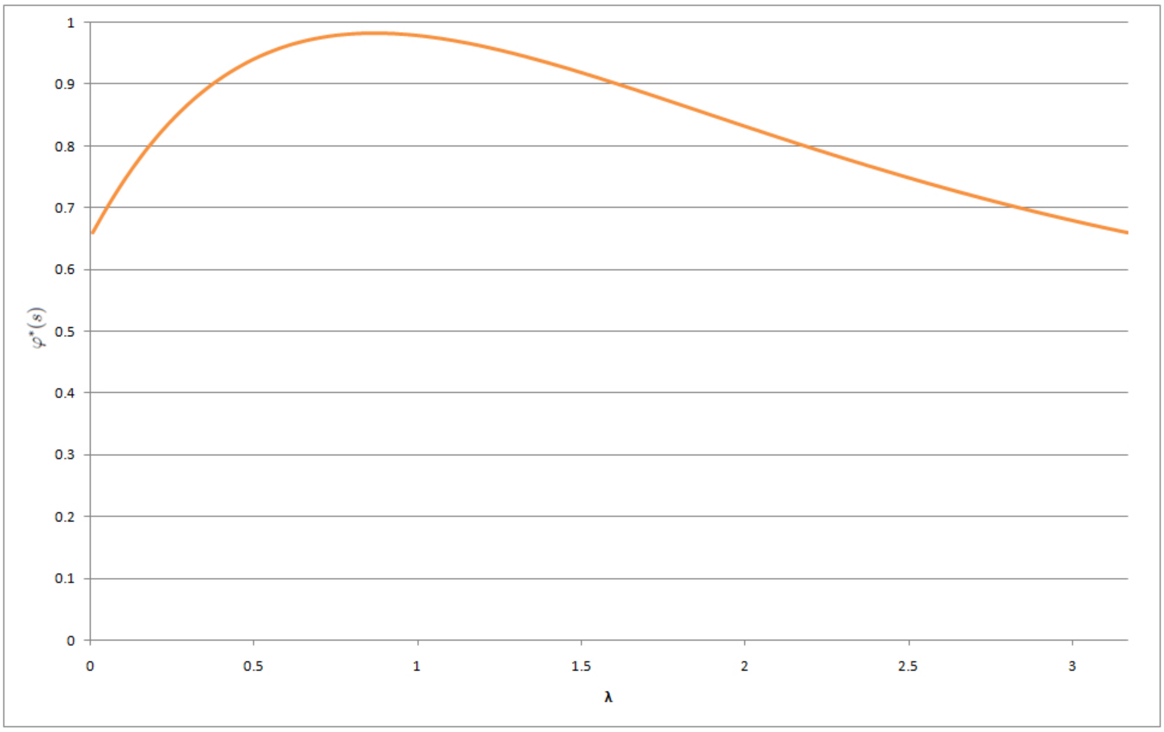

Figure 4.6: A graph demonstrating the relationship of $\varphi^*(s)$ and λ

volatility and increases their holdings of risky assets. Then as the more λ increases, the less realistic volatility becomes, the investor foresees greater risk and reduces his proportion of risky assets.

Chapter 5

Numerical Analysis Infinite Horizon

In this chapter, we aim to solve numerically the infinite horizon PDE derived in Chapter 3. While this chapter essentially applies the same substantive techniques of the previous chapter, it is worth examining separately as we are optimising over consumption as well.

Like the previous chapter, we will again base our parameters on the Dothan model. $\sigma(s)$ and $\mu(s)$ will be defined as in equations (4.2) and (4.3). With these choices equation (3.19) becomes:

$$\Psi(s) = \frac{(4\lambda + \xi^2 s)^2 \gamma}{8\xi^2} + \frac{r\gamma - \rho}{\gamma} \tag{5.1}$$

5.1 Monte Carlo Simulations

Using similar methods as the last chapter, I will numerically solve equation (3.24). We have used parameters $\gamma = 0.5$, $\xi = 0.5$, $\lambda = 0.5$, $\rho = 0.05$,$r = 0.05$, $N_1 = 25$ and $N_2 = -25$.

Notice that Figure (5.1) is a replica of Figure (4.3). This was what we expected because in Figure (4.3) we fixed time. This replicates Merton's observation [1], where:

$$\lim_{T\to\infty} [\varphi^*_{\text{finite horizon}}] = \varphi^*_{\text{infinite horizon}} \tag{5.2}$$

As our plots involving φ are identical to the last chapter, I will only

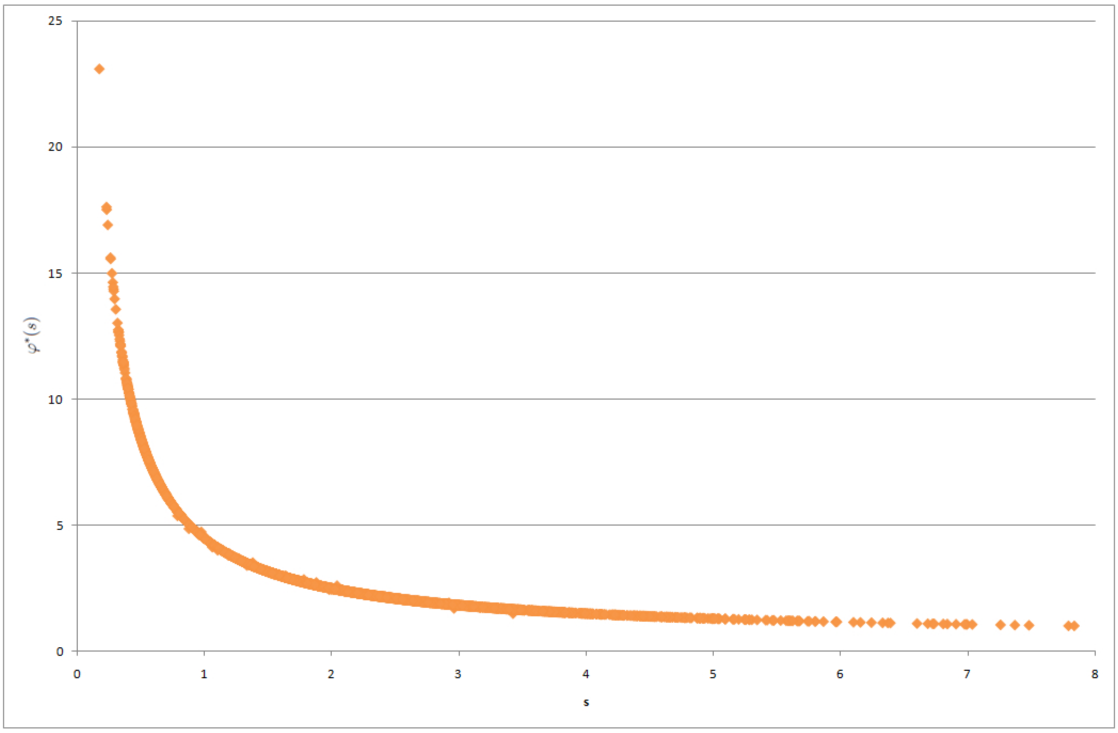

Figure 5.1: A scatter plot to demonstrate the relationship of $\varphi^*(s)$ and the offsets in the infinite time case.

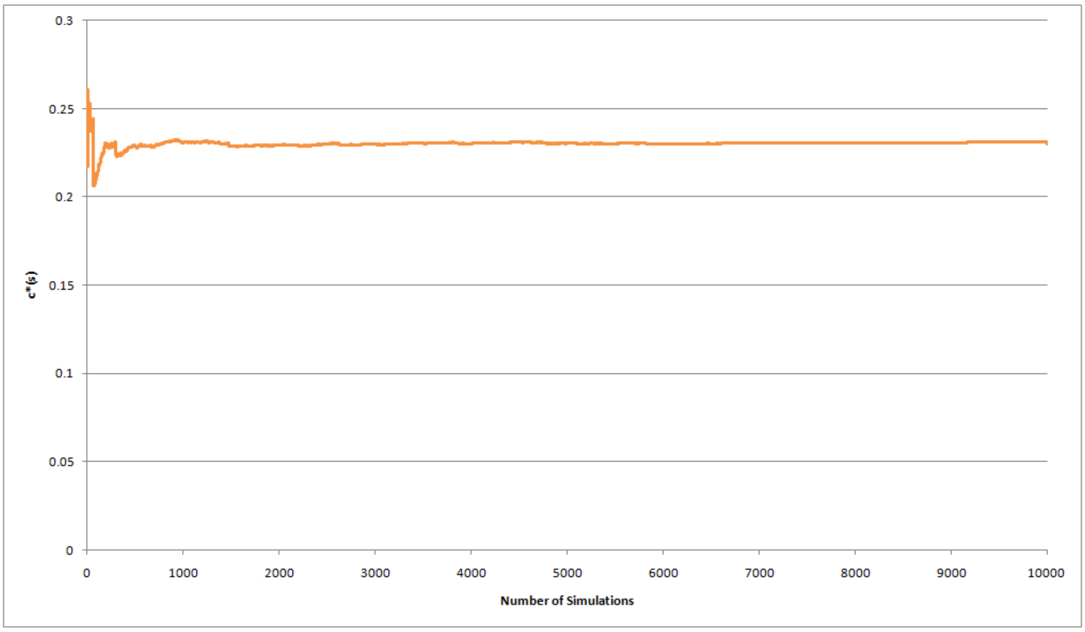

Figure 5.2: A plot to demonstrate the evolution of $c^*(s)$

focus on the optimal consumption in this chapter. As before, we will vary each parameters and discuss how they affect c^*. First of all we need to simplify equation (3.9). Looking at equations (3.15) and (3.22) we can reduce equation (3.9) to:

$$c^* = \gamma^{-\frac{1}{1-\gamma}} \frac{h}{n} \tag{5.3}$$

Thus taking $h = 1$, we can numerically approximate c^*. We have plotted this function in Figure (5.2). This graph describes the evolution of c^* as we progress through each simulation.

5.2 The Parameters

As in the previous chapter, we will discuss the parameters γ, ξ and λ and observe the effects varying them has on c^*. The c^* we have used here has been derived using 10000 simulations.

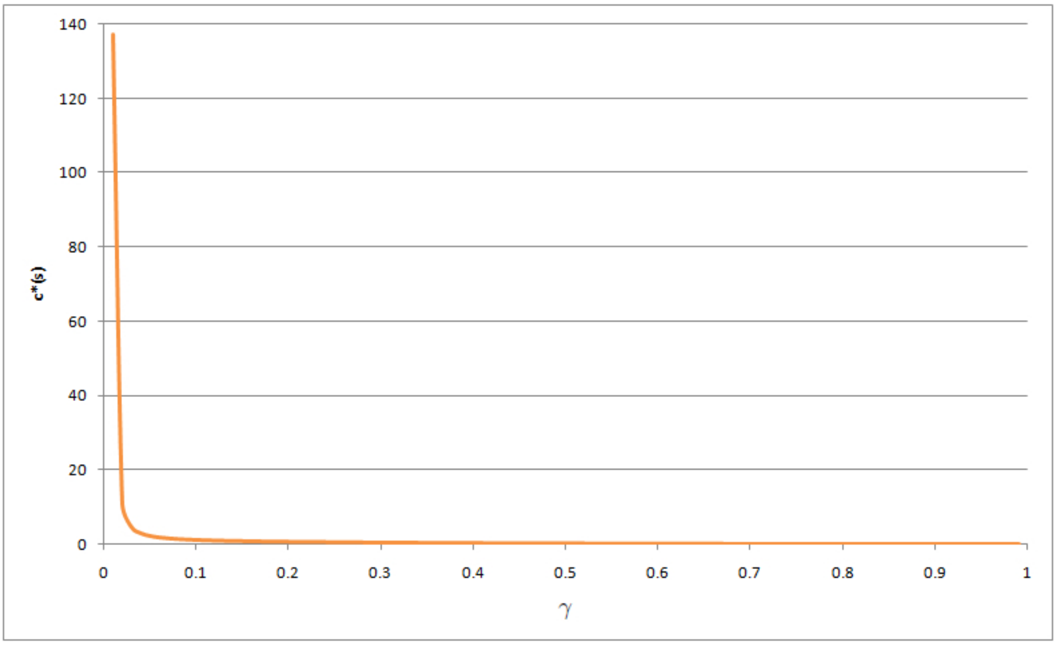

Figure 5.3: A plot to demonstrate the evolution of $c^*(s)$ as we vary γ.

5.2.1 The Risk Aversion Constant γ

In Figure (5.3), we demonstrate the relationship of $c^*(s)$ as we vary γ. In our case we have taken the parameters $\xi = 0.5$, $\lambda = 0.5$, $r = 0.05$, $\rho = 0.05$, $N_1 = 25$, $N_2 = -25$.

This plot is what we expected. Remember as $\gamma \to 1$ the more risks the investor takes. As the risk aversion constant increases in our plot, consumption rates fall. The investor who likes taking risks gets satisfaction from accumulating wealth. Thus, this sort of investor keeps consumption to a minimum, in order to amass the most wealth. This is clearly visible in Figure (5.3).

5.2.2 The Volatility Parameter ξ

In Figure (5.4), we demonstrate the relationship of $c^*(s)$ as we vary ξ. We have again taken the usual parameters: $\gamma = 0.5$, $\lambda = 0.5$, $r = 0.05$, $\rho = 0.05$, $N_1 = 25$, $N_2 = -25$.

We must remember that $c^*(s)$ is the optimal rate of consumption. When we look at Figure (5.4), we can clearly see that as the volatility increases the

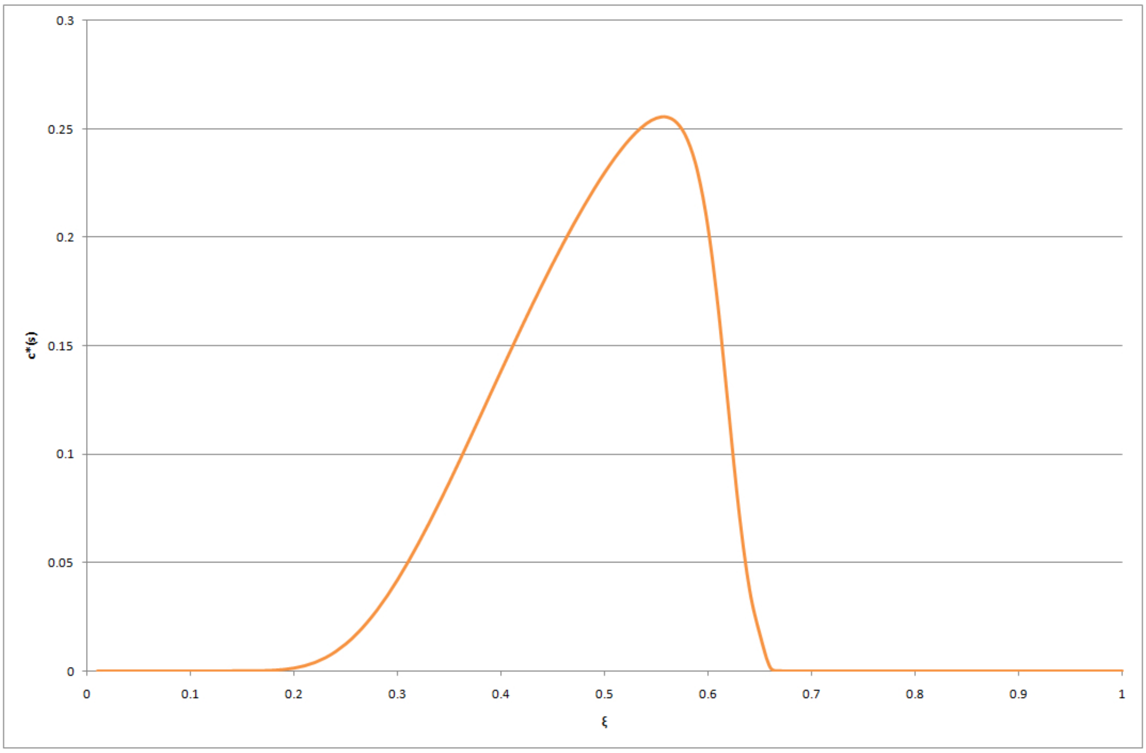

Figure 5.4: A plot to demonstrate the evolution of $c^*(s)$ as we vary ξ.

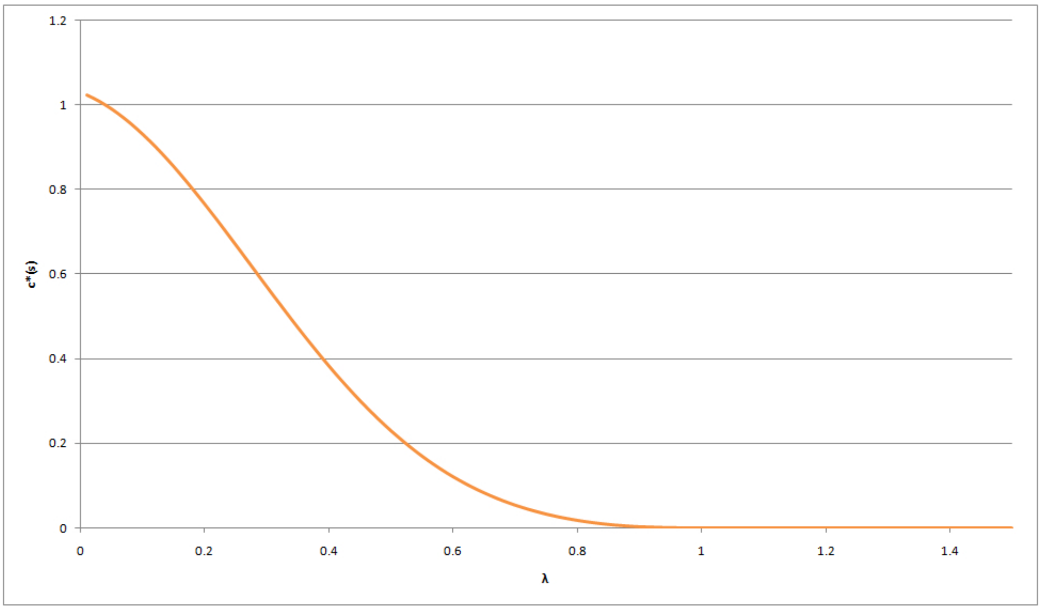

Figure 5.5: A plot to demonstrate the evolution of $c^*(s)$ as we vary λ.

rate of consumption initially increases too. However as soon as the volatility reaches a certain level the consumption falls sharply. This is what we expected; the initial increase in consumption was due to the investor not being aware at first of the approaching market conditions. At the time of the initial increase, the volatility was not high, thus not a cause for concern. Once the investor realises, we can see a sharp drop in consumption.

5.2.3 The Discounting Rate of Past Information λ

In Figure (5.5) we have plotted of $c^*(s)$ and λ with the parameters: $\gamma = 0.5$, $\xi = 0.50$, $r = 0.05$, $N_1 = 25$ and $N_2 = -25$.

As discussed before we know that lambda is the rate of discounting of past information. The lower the λ, the more reliable the model as the dynamics are based on recent stock prices. In figure (5.5), we find that as λ increases the consumption rate decreases. This is because initially the investor felt more comfortable consuming when he had a more reliable outlook on the market. However as λ increased the outlook for the investor became more unreliable, hence the investor reduced consumption.

Chapter 6

Conclusion

In this paper, we solved Merton's portfolio problem in a stochastic volatility environment. Previous work have been done on this topic, for example Fouque, Papanicolaou & Sircar [8] and Benth, Karlsen & Reikvam [19] tackled the problem when the volatility was explicitly driven by the Ornstein-Uhlenbeck process. However, this 'multi-factor' model had one major disadvantage over its 'single-factor' counterpart; it left the market incomplete. Thus, we decided to abandon the 'multi-factor' approach and focus on 'single-factor' models. After a basic review of the straightforward Level Dependent Volatility 'single-factor' models; we deduced that there were more advantages in choosing a model that defined volatility in terms of historical prices [18]. One such model was the complete stochastic volatility model of Hobson & Rogers [5]. With this framework, we were able to capture the well-known phenomena of *Volatility Persistence* and the *Leverage Effect*. This further reinforced the relationship between volatility and stock prices, both past and present.

Inspired by the work of Merton [1], Duffie, [7], Musiela & Zariphopoulou [9] we reduced the Merton's portfolio problem for both finite and infinite horizon cases into expectations. We then compared the parameters $\sigma(s)$ and $\mu(s)$ to the Dothan model. We chose the Dothan framework as it was the simplest model that did not violate the Lipschitz and positivity conditions required in the Hobson & Rogers model. Using Monte Carlo techniques, we approximated the expectations. In the finite horizon case we were able to deduce the value of the optimal fraction of wealth invested in the risky asset φ as a function of time and offsets. The drawbacks of this approach first emerged when we plotted the function against the individual parameters γ, λ and ξ. We must remember that $\varphi < 1$, meaning that we had to add further restrictions on our parameters. From a practitioners viewpoint this severely limits application, especially the restriction on ξ. This will mean we need to be aware of the volatility, which is not direct observable. Ideally we would like to permit all possible choices of volatility by forcing $\xi > 0$. However, this can never happen as the limits we took in equation (4.21) always means that for some values of ξ, φ will always be above 1. Nevertheless, the most preferable scenario would be to avoid all these conditions by simply resorting to another model instead of using the Dothan framework. A suitable choice would be the parameters Hobson & Rogers defined in [5].

$$\sigma s = \nu\sqrt{1 + \epsilon s^2} \wedge N \tag{6.1}$$

for some large constant N. ν and ϵ are parameters we will need to fit. A suitable choice of drift $\mu(s)$ will make the offset mean reverting; this property will be inherited by $\sigma(s)$. Then instead of generating random variables, we calibrate the parameters on historical data of actual stock prices. This should avoid the problems we had with the parameters above. This option is left as an idea for further research.

Let us for the moment only consider the $0 < \varphi < 1$ part on the plots of each parameter. In the relationship of φ and the risk aversion constant, we see the value of γ is proportional correlated to φ. Again, makes sense, as the risk nature of the investor will affect his position in the risky asset. In the relationship of φ and λ we noticed that there was a slight hump in the plot. This made sense for smaller λ, an investor would usually find the model more accurate than for a larger λ. This is because when you consider a larger λ means we discount more; which in turn implies that in our offset we considered historical stock prices where one was taken after significant period of time than the other. This meant the offset function had a sort of dampening effect where many of the shocks in between were in a sense smoothed out. This leads the investor to rebalance their portfolio, decreasing their proportion of risky assets. This sounds eccentric, as in our case we only considered one offset, but over multiple offsets we will be able to get more accurate representations of the dynamics. This can be another suggestion for further research. However, one must be aware that the optimality equations involve a PDE, considering more offsets would modify the problem to solving a higher dimension PDE, which could prove quite difficult to solve.

Finally, we considered the problem in the infinite time horizon setting. In this scenario, we considered consumption and replaced the boundary condition (2.21) with a so-called *transversality condition*. Reminiscent of Merton [1] we ended up with the same φ as the finite horizon case. We tested the effects of the parameters on the optimal consumption c^*. In the relationship of c^* and the risk aversion constant we clearly saw that as γ increased consumption decreased. This is one of the traits of a risk loving investor. They will consume less in order to maximise wealth. In the relationship of c^* and the volatility parameter we clearly saw that as ξ was a hump shape, indicating an all of a sudden realisation of market conditions by the investor. This trait occurs frequently in practice. In the relationship of c^* and the λ we clearly saw that the investor only liked to consume when he knew he was likely to make back some wealth in the markets.

Appendix A

Appendix

A.1 Monte Carlo Simulations, Normal.cpp

All C++ code have been tested on the Microsoft Visual C++ 2008 compiler running on a Microsoft Windows Vista 64bit system. Performance may vary from system to system. The C++ code for Linear Congruential Generators is slightly based on the one provided by Scheinerman [27]. This code implements the methods of Linear Congruential Generators and the Box-Muller Algorithm to generate sample paths for $\tilde{S}_t$.

```
/***
 * A simple program to generate Standard Normal Random Variables
 ***/

#include <iostream>
#include <cmath>
#include <vector>
#include <fstream>

using namespace std;

//Define Linear Congruential Generator
int lcg(long int a, long int b, long int c) {
static int state = 1; //seed
state = (a*state+b) % c;
if (state<0)
         return state+c;
```

```
    else
         return state;
}

int main() {
char name[5];
int Number_Of_Simulations = 100000;

ofstream out("Normal.txt");

const double PI = 3.1415926535897932; // Define Pi
vector<double> Uniform(100000);

//Park & Miller
long int a = 6807;
long int b = 0;
long int c = 2147483647;

// Generate Uniform Random Variables
for (int k=0; k<Number_Of_Simulations; k++) {
    Uniform[k] = (lcg(a,b,c) /2147483647.);
}
for(int i=0; i<Number_Of_Simulations;i+=2){
    //Define the Normal Random variables using Box Muller
     out << sqrt(-2*log(Uniform[i]))*cos(2*PI*Uniform[i+1]) << endl;
     out << sqrt(-2*log(Uniform[i]))*sin(2*PI*Uniform[i+1]) << endl;
}

Uniform.clear(); //Free up memory by erasing this vector
cout << "We are generating "<<Number_Of_Simulations
     <<" Standard Normal Random Variables" << endl;
cout << "The result is being written to Normal.txt" << endl;
cout << "Hit any key+<RET> to finish" << endl;
cin  >> name;

return 0;
}
```

This program outputs to a text file, after which we load it up onto Microsoft Excel and perform Monte Carlo Simulations to approximate the expectation we obtained at the end of chapter 2 and 3.

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